

DISCRETE MORSE THEORY FOR MOMENT-ANGLE COMPLEXES OF PAIRS (D^n, S^{n-1})

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ABSTRACT. For a finite simplicial complex K and a CW-pair (X, A) , there is an associated CW-complex $\mathcal{Z}_K(X, A)$, known as a polyhedral product. We apply discrete Morse theory to a particular CW-structure on the n -sphere moment-angle complexes $\mathcal{Z}_K(D^n, S^{n-1})$. For the class of simplicial complexes with vertex-decomposable duals, we show that the associated n -sphere moment-angle complexes have the homotopy type of wedges of spheres. As a corollary we show that a sufficiently high suspension of any restriction of a simplicial complex with vertex-decomposable dual is homotopy equivalent to a wedge of spheres.

1. INTRODUCTION

In this paper we study the following construction of topological spaces. Let K be a simplicial complex K over ground set $[m] := \{1, \dots, m\}$ and let (X, A) be a pair of spaces. For a simplex $\sigma \in K$ set

$$(X, A)^\sigma := \{(x_1, \dots, x_m) \in X^m \mid x_i \in A, i \notin \sigma\}.$$

and define $\mathcal{Z}_K(X, A) \subseteq X^m$ as

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma$$

equipped with the subspace topology of the Cartesian product X^m . The space $\mathcal{Z}_K(X, A)$ is called the *polyhedral product* of K with respect to the pair (X, A) .

Polyhedral products are a natural generalization of Cartesian products and appear prominently in various areas of mathematics (see [4], [14], [15], [13], [19] and [1] for references and details). Despite its simple combinatorial definition the topology of $\mathcal{Z}_K(X, A)$ is hard to control. The most far reaching general results on the homotopy type can be found in [15] where a wedge decomposition in terms of the homotopy type of induced subcomplexes of K is given for shifted simplicial complexes K and pairs (CX, X) of a cone CX over X , and in [13] where among others it is proved that for simplicial complexes K whose non-faces form a chordal graph $\mathcal{Z}_K(D^2, S^1)$ has the homotopy type of a wedge of spheres. Of particular interest is the case when $(X, A) = (D^n, S^{n-1})$ for some number $n \geq 1$, where D^n is the n -dimensional disk and S^{n-1} its bounding $(n-1)$ -sphere. Among

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those the case $n = 2$ plays the most important role since it relates to torus action (see [4]) and coordinate subspace arrangements (see Section 2). Extending standard terminology from the case $n = 2$ we call the space $\mathcal{Z}_K(D^n, S^{n-1})$, the *n-sphere moment-angle complex*.

We extend the result from [15] (see also [19]) in the case $(X, A) = (D^n, S^{n-1})$ from the class of shifted simplicial complexes to the class of simplicial complexes K for which its Alexander dual $K^\circ := \{\sigma \subset [m] \mid [m] \setminus \sigma \notin K\}$ is vertex decomposable. We also refer the reader to Section 2.1 for unexplained terminology. Our main result states:

Theorem 1.1. *Let K be a simplicial complex on ground set $[m]$ and $\{i\} \in K$ for all $i \in [m]$ with vertex decomposable Alexander dual complex K° . Then the n -sphere moment angle complex $\mathcal{Z}_K(D^n, S^{n-1})$ has a homotopy type of the wedge of spheres*

$$(1) \quad \mathcal{Z}_K(D^n, S^{n-1}) \simeq \bigvee_{i \geq 0} \bigvee_{\substack{M \not\supset K \\ M \subseteq [m]}} \dim_{\mathbb{K}} \tilde{H}_{i-(n-1)\#M-1}(K_M, \mathbb{K}) \cdot S^i$$

for any field \mathbb{K} , where K_M is the restriction of K to M .

The intuition for the theorem comes from a result by Buchshtaber and Panov [4, Thm. 7.7]. There they prove that the cohomology ring of $\mathcal{Z}_K(D^2, S^1)$ is isomorphic to the Tor-algebra of the Stanley-Reisner ring of the simplicial complex K extending an additive isomorphism from [12]. From [17] we know that if K° is a (non-pure) shellable simplicial complex (see [5]) and the Stanley-Reisner ideal of K is generated in degrees ≥ 2 (i.e. $\{i\} \in K$ for all $i \in [m]$) then the Stanley-Reisner ring of K is Golod over all fields. The Golod property [16] says that all Massey operation on the Koszul complex vanish – for which the first is the product on the Tor-algebra. By the work of Berglund and Jöllenbeck [2] for Stanley-Reisner rings the Golod property actually is equivalent to the vanishing of the product on the Tor-algebra. Since vertex-decomposable simplicial complexes are shellable [6, Thm. 11.3] and hence sequentially Cohen-Macaulay it follows from [17, Thm. 4] that the product on the cohomology ring of $\mathcal{Z}_K(D^2, S^1)$ must be trivial if K° is vertex-decomposable. Indeed in [3, Thm. 6] it is proved that $\mathcal{Z}_K(D^2, S^1)$ is rationally homotopy equivalent to a wedge of spheres if and only if K the rational Stanley-Reisner ring of K is Golod. But this provides insight only for the case $n = 2$. The fact that $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres for all $n \geq 1$ for K with vertex-decomposable dual remains as mysterious as the previous results from [15] for the subclass of shifted simplicial complexes and pairs (CX, X) , where CX is the cone over X . Nevertheless, the homotopy equivalence from Theorem 1.1 is known to hold in the stable category using a single suspension only. It follows from [1, Cor. 2.24] (see Proposition 2.6) below) that for any K and any (X, A) the suspension of $\mathcal{Z}_K(X, A)$ has a homotopy type given by the suspension of formula from Theorem 1.1.

A simple calculation shows that the Alexander dual of a shifted simplicial complex is also shifted. Thus by [6, Thm. 11.3] shifted simplicial complexes have vertex decomposable Alexander dual complexes. Hence the result of Theorem 1.1 contains the main result of [15] for the important case $(CX, X) = (D^n, S^{n-1})$. For the subclass of skeleta K of simplices

it was previously shown in [14] that the moment-angle complexes $\mathcal{Z}_K(D^2, S^1)$ have the homotopy type of wedges of spheres.

Finally, using a result by Eagon and Reiner [9, Prop. 8] we obtain as a corollary the following slight extension of a result from [13], which is the main new result of their Theorem 4.6. For its formulation we denote by $K^{(1)}$ the 1-skeleton of the simplicial complex K which in turn can be considered a graph. Now $K^{(1)}$ is called *chordal* if all cycles of length ≥ 4 have a chord. Recall that a simplicial complex K is called *flag* if all its minimal non-faces are of size 2.

Corollary 1.2 (see Theorem 4.6 [13]). *Let K be a flag simplicial complex such that $K^{(1)}$ is a chordal graph. Then $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres.*

We note that [9, Prop. 7] implies that if the minimal non-faces of K are supported on the bases of a matroid then K° is vertex-decomposable. Similarly, the work of Billera and Provan [20] shows that quite a large class of those simplicial complexes for which K° is the boundary complex of a simplicial polytope satisfies the assumptions of Theorem 1.1. Thus in either case we obtain that $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres.

In contrast to [14] and [15], where more advanced tools from homotopy theory are invoked, our methods are combinatorial and elementary. We apply discrete Morse theory [10] to a suitably chosen regular cell decomposition and some basic lemmas from homotopy theory. Indeed, we show in Corollary 4.6 (iii) that the critical cells arising from our application of discrete Morse theory all contribute a sphere. Since by Proposition 2.5 the space $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a complement of a subspace arrangement this shows that our application of discrete Morse theory yields results that are similar in spirit to the ones in [21] and [8]. There a cell decomposition for complements of arbitrary complex hyperplane arrangement is constructed such that the cells form a basis of homology.

Finally, as an interesting consequence in geometric combinatorics we obtain the following corollary of Theorem 1.1.

Corollary 1.3. *Let K be a simplicial complex on ground set $[m]$ with vertex decomposable Alexander dual complex K° . Then for any $M \subseteq [m]$ the $(\#M + 2)$ -fold suspension $\Sigma^{\#M+2}K_M$ of the restriction K_M of K to M is homotopy equivalent to a wedge of spheres.*

The paper is organized as follows. In Section 2, we provide basic facts about vertex decomposable simplicial complexes and polyhedral products. In particular, we describe the relation of polyhedral products and coordinate subspace arrangements. In Section 3, we first review Forman's discrete Morse theory [10] in terms of acyclic matching. Then for an arbitrary simplicial complex K , we construct the matching \mathcal{M}_K corresponding to an n -sphere polyhedral product complex $\mathcal{Z}_K(D^n, S^{n-1})$ and identify the set of critical cells $\text{Crit}(\mathcal{M}_K)$. In Section 4, we use the results from Section 3 to complete the proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3.

2. BASICS ABOUT VERTEX DECOMPOSABLE COMPLEXES AND POLYHEDRAL PRODUCTS



FIGURE 1. Simplicial Complex and its Dual

2.1. Vertex Decomposable Simplicial Complexes. First we recall some basic terminology for simplicial complexes. Let $K \subseteq 2^{[m]}$ be a simplicial complex over the ground set $[m]$. The elements of K are called *faces* and the inclusionwise faces are called *facets* of K . For $M \subseteq [m]$ we write $K_M := \{\sigma \in K \mid \sigma \subseteq M\}$ for the *restriction* of K to M . For $v \in [m]$ we denote by $K \setminus v := K|_{[m] \setminus \{v\}}$ the *deletion* of v and by $\text{link}_K(v) = \{\sigma \in K \mid v \notin \sigma, \sigma \cup \{v\} \in K\}$ its *link*. We consider the link $\text{link}_K(v)$ and the deletion $K \setminus v$ of K at a vertex v as simplicial complexes over the ground set $[m] \setminus \{v\}$. For two simplicial complexes K_1 and K_2 over disjoint ground sets we write $K_1 * K_2 := \{A_1 \cup A_2 \mid A_i \in K_i, i = 1, 2\}$ for their *join*. Using this terminology the (closed) *star* $\text{star}_K(v) := \{\sigma \in K \mid \sigma \cup \{v\} \in K\}$ of a vertex v can be written as $\text{star}_K(v) = \{\emptyset, \{v\}\} * \text{link}_K(v)$.

The notion of a vertex decomposable simplicial complex was defined for pure complexes in [20] and extended to nonpure complexes in [6]. The class of vertex decomposable simplicial complexes is defined inductively. A simplicial complex K is called *vertex decomposable* if:

- (i) K is a simplex or $K = \{\emptyset\}$, or
- (ii) there exists a vertex v such that:
 - (a) $K \setminus v$ and $\text{link}_K(v)$ are vertex-decomposable
 - (b) no facet of $\text{link}_K(v)$ is a facet of $K \setminus v$.

The distinguished vertex v in (ii) is called a *shedding vertex*. We say that an ordering $(v_1, v_2, \dots, v_\ell)$ of some of the vertices of a simplicial complex K is a *shedding sequence* if v_k is a shedding vertex of the consecutive deletion $K \setminus v_\ell \setminus \dots \setminus v_{k+1}$ for all $k = 1, 2, \dots, \ell - 1$ and $K \setminus v_\ell \setminus \dots \setminus v_1$ is a simplex.

Example 2.1. The simplicial complex $K \subseteq 2^{[4]}$ with facets $\{1, 3\}$ and $\{2, 4\}$ has the vertex decomposable Alexander dual K° with facets $\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}$. In this case $(1, 2, 3, 4)$ is a shedding sequence for K° , while the sequence $(1, 3, 2, 4)$ is not (see Figure 2.1).

We are interested in the class of simplicial complexes with vertex decomposable Alexander duals. First we list a some well known facts that we will use in subsequent arguments.

Remark 2.2. Let K be a simplicial complex on ground set $[n]$.

- (i) Then $N \subseteq [n]$ is a minimal non-face of K if and only if $[n] \setminus F$ is a facet of K° .
- (ii) For $v \in [n]$ we have $\text{link}_K(v)^\circ = K^\circ \setminus v$ and $(K \setminus v)^\circ = \text{link}_{K^\circ}(v)$.

The definition of vertex decomposability and [Remark 2.2](#) immediately imply the following lemma.

Lemma 2.3. *Let K be a simplicial complex with vertex decomposable Alexander dual K° . If v is a shedding vertex of K° then the link $\text{link}_K(v)$ and the deletion $K \setminus v$ have vertex decomposable duals.*

2.2. Polyhedral Products. Let us first collect some basic facts about polyhedral products that we will use in subsequent sections without explicit reference.

- ◊ If $L \subset K$ is a subcomplex over the same ground set then $\mathcal{Z}_L(X, A)$ a subspace of $\mathcal{Z}_K(X, A)$.
- ◊ A map $f : (X, A) \longrightarrow (Y, B)$ gives rise to a map $\mathcal{Z}_K(f) : \mathcal{Z}_K(X, A) \longrightarrow \mathcal{Z}_K(Y, B)$. For the composition of induced maps we have $\mathcal{Z}_K(g \circ f) = \mathcal{Z}_K(g) \circ \mathcal{Z}_K(f)$.
- ◊ $\mathcal{Z}(\bullet, \bullet)$ is the homotopy functor; that is the homotopy type of $\mathcal{Z}_K(X, A)$ depends only on the relative homotopy type of the pair (X, A) .

Next we stress the relevance of the spaces $\mathcal{Z}_K(D^n, S^{n-1})$ by exhibiting a homotopy equivalence to $\mathcal{Z}_K(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, which in turn is easily seen to be the complement of a suitably chosen coordinate subspace arrangement in $(\mathbb{R}^n)^m$. Clearly, $\mathcal{Z}_K(\mathbb{C}, \mathbb{C} \setminus \{0\}) = \mathcal{Z}_K(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\})$ is the complement of the vanishing locus of the Stanley-Reisner ideal of K° in \mathbb{C}^m .

Before we can show that $\mathcal{Z}_K(D^n, S^{n-1})$ and $\mathcal{Z}_K(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are homotopy equivalent we need one elementary set theoretic fact.

Lemma 2.4.

$$\mathcal{Z}_{K^\circ}(X, A) = X^m \setminus \mathcal{Z}_K(X, X \setminus A).$$

Proof. For a $x = (x_1, \dots, x_m) \in X^m$ denote by $I_x = \{i \in [m] \mid x_i \in A\}$. We have that $x \in \mathcal{Z}_{K^\circ}(X, A)$ if and only if $\sigma^c \subset I_x$ for some $\sigma \in K^\circ$ and $x \notin \mathcal{Z}_K(X, X \setminus A)$ if and only if $I_x \cap \sigma^c \neq \emptyset$ for all $\sigma \in K$. Both statements are equivalent to $I_x \notin K$. \square

The coordinate subspace arrangement $\mathcal{A}_{K,n}$ associated to the complex K is defined as the K° -power of the pair $(\mathbb{R}^n, \{0\})$

$$\mathcal{A}_{K,n} = \mathcal{Z}_{K^\circ}(\mathbb{R}^n, \{0\}) = \bigcup_{\sigma \in K^\circ} (\mathbb{R}^n, \{0\})^\sigma.$$

By [Lemma 2.4](#), the complement $\mathcal{A}_{K,n}^c = (\mathbb{R}^n)^m \setminus \mathcal{A}_{K,n}$ of the arrangement $\mathcal{A}_{K,n}$ is determined as the polyhedral product of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$

$$\mathcal{A}_{K,n}^c = \mathcal{Z}_K(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).$$

Since $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ and $(D^n, D^n \setminus \{0\})$ are relative homotopy equivalent and $\mathcal{Z}_K(X, A)$ is a homotopy functor, it follows that $\mathcal{A}_{K,n}^c$ and $\mathcal{Z}_K(D^n, D^n \setminus \{0\})$ have the same homotopy type. This type of reasoning does not suffice to prove the homotopy equivalence of $\mathcal{Z}(D^n, D^n \setminus \{0\})$ and $\mathcal{Z}(D^n, S^{n-1})$ since $(D^n, D^n \setminus \{0\})$ and (D^n, S^{n-1}) are not homotopy equivalent as pairs. Indeed Jelene Grbić pointed out to us that $\mathcal{Z}_K(\bullet, \bullet)$ is a homotopy functor in both arguments from which the results follows easily. For the sake of an explicit homotopy we still provide the proof. For $n = 2$ an explicit homotopy equivalence of $\mathcal{Z}(D^2, D^2 \setminus \{0\})$ and $\mathcal{Z}(D^2, S^1)$ was established in [4]. We provide the homotopy equivalence for any $n \geq 1$.

Proposition 2.5. *For $n \geq 1$ the complement $\mathcal{A}_{K,n}^c$ of the arrangement $\mathcal{A}_{K,n}$ is homotopy equivalent to the n -sphere moment-angle complex $\mathcal{Z}_K(D^n, S^{n-1})$.*

Proof. A simple retraction argument shows that the complex $\mathcal{A}_{K,n}^c = \mathcal{Z}_K(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ is homotopy equivalent to $\mathcal{Z}_K(D^n, D^n \setminus \{0\})$. We inductively construct another deformation retraction

$$r_K : \mathcal{Z}_K(D^n; D^n \setminus \{0\}) \rightarrow \mathcal{Z}_K(D^n; S^{n-1}).$$

Our induction basis is the case when $K = 2^{[k]} \subseteq 2^{[m]}$ is a full k -simplex. In this case we have $\mathcal{Z}_K(D^n; D^n \setminus \{0\}) = D^{nk} \times (D^n \setminus \{0\})^{m-k}$ and $\mathcal{Z}_K(D^n; S^{n-1}) = D^{nk} \times (S^{n-1})^{m-k}$. Then the identity on the coordinates $[k]$ and the radial projection $\pi_n : D^n \setminus \{0\} \mapsto S^{n-1}$ in the coordinates $\{k+1, \dots, m\}$ provides a deformation retraction from $r_K : \mathcal{Z}_K(D^n; D^n \setminus \{0\}) \rightarrow \mathcal{Z}_K(D^n; S^{n-1})$.

In the induction step $K = K' \cup \{\tau\}$ for some facet τ of K . We may assume $\tau = [k]$ after suitable relabeling. In this situation

$$\mathcal{Z}_{K'}(D^n; D^n \setminus \{0\}) = \mathcal{Z}_K(D^n; D^n \setminus \{0\}) \setminus \{(0, \dots, 0) \times (D^n \setminus \{0\})^{m-k}\}$$

and

$$\mathcal{Z}_{K'}(D^n; S^{n-1}) = \mathcal{Z}_K(D^n; S^{n-1}) \setminus (D^n \setminus S^{n-1})^k \times (S^{n-1})^{m-k}$$

Case 1: $K = 2^{[k]}$ is a full $(k-1)$ -simplex and $\tau = [k]$.

Then

$$\mathcal{Z}_{K'}(D^n; D^n \setminus \{0\}) = (D^{nk} \setminus \{(0, \dots, 0)\}) \times (D^n \setminus \{0\})^{m-k}$$

and

$$\mathcal{Z}_{K'}(D^n; S^{n-1}) = S^{mk-1} \times (S^{n-1})^{m-k}.$$

Hence the projection $D^{nk} \setminus \{(0, \dots, 0)\} \mapsto S^{mk-1}$ on the first k coordinates and the projection π_n in each of the remaining coordinates yields a deformation retraction $r_{K'}$.

Case 2: $\tau = [k]$ is not the unique facet of K .

In this case we can write K as the union of K' and $2^{[k]}$ with intersection $L = 2^{[k]} \setminus \{[k]\}$. By induction for K and by Case 1 for L we have deformation retraction maps $r_K : \mathcal{Z}_K(D^n; D^n \setminus \{0\}) \rightarrow \mathcal{Z}_K(D^n; S^{n-1})$ and $r_L : \mathcal{Z}_L(D^n; D^n \setminus \{0\}) \rightarrow \mathcal{Z}_L(D^n; S^{n-1})$. Let $i : \mathcal{Z}_{K'}(D^n; D^n \setminus \{0\}) \hookrightarrow \mathcal{Z}_K(D^n; D^n \setminus \{0\})$ be the inclusion map. Then consider

$r := r_L \circ r_K \circ i$. Then the image of r is $\mathcal{Z}_{K'}(D^n; S^{n-1})$ and a simple computation shows that $r_{K'} = r$ is the desired retraction. \square

Complements of subspace arrangements are an interesting object for their own sake and their homotopy types are usually difficult to approach.

We finally recall a result from [1] on the homotopy type of the suspension $\Sigma \mathcal{Z}_K(D^n, S^{n-1})$ of $\mathcal{Z}_K(D^n, S^{n-1})$, that by Proposition 2.5 also gives a homotopy decomposition of the suspension of the subspace arrangement complement $\mathcal{A}_{K,n}^c$.

Proposition 2.6 (Cor. 2.24 in [1]). *For $n \geq 2$ there is a homotopy equivalence*

$$\Sigma \mathcal{Z}_K(D^n, S^{n-1}) \simeq \bigvee_{\substack{M \not\subseteq K \\ M \subseteq [m]}} \Sigma K_M * S^{(n-1)\#M}.$$

3. DISCRETE MORSE THEORY FOR $\mathcal{Z}_K(D^n, S^{n-1})$

In this section we exhibit the tools and perform the basic constructions needed for the proof of Theorem 1.1.

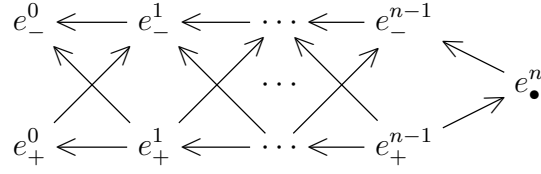
Let X be a compact regular CW-complex. It is well known [11, Prop. 1.2.] that for regular CW-complexes the topology of X is determined by its face poset; that is the partial order \preceq on the set $X^{(\bullet)}$ of closed cells of X ordered by $c \preceq c'$ if and only if $c \subseteq c'$. The directed graph $G_X = (V_X, E_X)$ on vertex set $V_X = X^{(\bullet)}$ and edge set

$$E_X = \{c \rightarrow c' \mid c, c' \in X^{(\bullet)}, c \succeq c', \dim(c) = \dim(c') + 1\}$$

is the graph of the Hasse diagram of $X^{(\bullet)}$. An acyclic matching on G_X is a set $\mathcal{M} \subseteq E_X$ of edges of G_X such that:

- ◇ Each cell from $X^{(\bullet)}$ appears in at most once edge from \mathcal{M}
- ◇ The directed graph $G_X^{\mathcal{M}} = (V_X, E_X^{\mathcal{M}})$ with edge set $E_X^{\mathcal{M}} = E_X \setminus \mathcal{M} \cup \{c' \rightarrow c \mid c \rightarrow c' \in \mathcal{M}\}$ contains no directed cycles.

Note that $G_X^{\mathcal{M}}$ arises from G_X by reversing all edges from \mathcal{M} . Discrete Morse theory was developed by Forman [10] in order to explicitly determine the homotopy type of regular CW-complexes. In order to formulate his main result we need some more notation. For an acyclic matching \mathcal{M} in G_X we call a cell $c \in X^{(\bullet)}$ an \mathcal{M} -critical cell if c does not appear in any edge from \mathcal{M} . We write $\text{Crit}(\mathcal{M})$ for the critical cells of the matching \mathcal{M} . For an i -cell c denote by f_c the attaching map $f_c : \partial c \rightarrow X^{i-1}$, where X^{i-1} is the $(i-1)$ -skeleton of X . To any edge $e : c \rightarrow c'$ in $G_X^{\mathcal{M}}$ we associate a map f_e . If $c \rightarrow c'$ is an edge from E_X then this map is the restriction $f_c|_{f_c^{-1}(c')}$ of the attaching map f_c of c to the preimage of c' . If $e : c \rightarrow c' \in E_X^{\mathcal{M}} \setminus E_X$ then f_e is an arbitrary but fixed homeomorphism from c to $\partial c'$ for which $f_{c'} \circ f_e(x) = x$ for all $x \in \partial c$. For two \mathcal{M} -critical cells c and c' we call a directed path $p : c = c_0 \rightarrow \cdots \rightarrow c_n = c'$ in $G_X^{\mathcal{M}}$ of length $n \geq 1$ a gradient path. To each gradient path we associate the map $f_p : \partial c \rightarrow c'$ which is the composition of the maps f_e corresponding to the edges of p . It is easy to see that the map f_p for all gradient paths starting in the i -critical cell c define a map $f_c^{\mathcal{M}}$ from ∂c to the set of \mathcal{M} -critical cells of dimension less than i . By induction one then shows that this defines a CW-structure on

FIGURE 2. The graph $G^{\mathcal{M}}$

the set $\text{Crit}(\mathcal{M})$ of \mathcal{M} -critical cells. We denote this CW-complex by $X^{\mathcal{M}}$. We are now in position to formulate the main result from Forman's discrete Morse theory using its reformulation in terms of acyclic matchings provided by Chari [7, Proposition 3.3].

Theorem 3.1 (Forman, Theorem 3.4 in [10]). *Let X be a compact regular CW-complex and let \mathcal{M} be an acyclic matching on G_X . Then X is homotopy equivalent to the CW-complex $X^{\mathcal{M}}$, where \mathcal{M} -critical cell $c \in \text{Crit}(\mathcal{M})$ of dimension i is attached in $X^{\mathcal{M}}$ to the $(i-1)$ -skeleton of $X^{\mathcal{M}}$ via the map $f_c^{\mathcal{M}}$.*

Let (X, A) be a regular CW-pair with the set of cells $V_A \subseteq V_X$, where $V_X = \{c_\lambda\}_{\lambda \in I}$ and $V_A = \{c_\lambda\}_{\lambda \in J}$ for some $J \subseteq I$. Then $\mathcal{Z}_K(X, A)$ is the CW-subcomplex of the product X^m with the set of cells

$$V_{\mathcal{Z}_K(X, A)} = \bigcup_{\sigma \in K} \{c_{\lambda_1} \times \cdots \times c_{\lambda_m} \mid \lambda_i \in J, i \notin \sigma\}.$$

Since any product of regular CW-complexes is regular [11, Theorem 2.2.2 (iv)] it follows that X^m carries a regular CW-structure and so does any subcomplex. In particular, the cellular structure on $\mathcal{Z}_K(X, A)$ is regular. The directed graph $G_{\mathcal{Z}_K(X, A)}$ is the subgraph of the product graph $G_{X^m} = (G_X)^m$.

In the following for a simplicial complex K and assuming a fixed pair (X, A) we write V_K for $V_{\mathcal{Z}_K(X, A)}$, E_K for $E_{\mathcal{Z}_K(X, A)}$ and G_K for $G_{\mathcal{Z}_K(X, A)}$.

The next lemma follows immediately from definitions.

Lemma 3.2. *If K_M is the restriction of the simplicial complex K on a subset $M \subseteq [m]$, then*

$$\mathcal{Z}_{K_M}(X, A) = (X, A)^M \cap \mathcal{Z}_K(X, A).$$

In the following, we assume that $(X, A) = (D^n, S^{n-1})$ for some $n \geq 1$, with the minimal regular cellular structure. That is, the sphere S^{n-1} has two cells e_-^i, e_+^i in each dimension i for $0 \leq i \leq n-1$ and the disc D^n has one additional cell e_\bullet^n of dimension n . Thus we have $V_{S^{n-1}} = \{e_-^i, e_+^i \mid i = 0, 1, \dots, n-1\}$ and $V_{D^n} = V_{S^{n-1}} \cup \{e_\bullet^n\}$. Let G be the graph of the corresponding Hasse diagram of the disc D^n and \mathcal{M} be the acyclic matching on G defined by

$$\mathcal{M} = \{e_-^{i+1} \rightarrow e_+^i \mid i = 0, 1, \dots, n-2\} \cup \{e_\bullet^n \rightarrow e_+^{n-1}\}.$$

The graph $G^{\mathcal{M}}$ is visualized in Figure 2.

We call the cells e_-^i minus-cells and the cells e_+^i plus-cells and consider minus or plus as their signs. Let $G_m = (V_m, E_m)$ be the graph of the Hasse diagram of the cell decomposition induced on the product $(D^n)^m$ with the vertex set $V_m = \{c_1 \times \cdots \times c_m \mid c_i \in V_{D^n}\}$. For a cell $c = c_1 \times \cdots \times c_m \in V_m$, define $\text{supp}(c) = \{i \in [m] \mid c_i = e_\bullet^n\}$. Let K be an arbitrary simplicial complex K over the ground set $[m]$. The graph G_K is the subgraph of G_m with the vertex set $V_K = \{c \in V_m \mid \text{supp}(c) \in K\}$. We construct the matching \mathcal{M}_K inductively. Let

$$\mathcal{M}_1 = \{c \rightarrow c' \in E_K \mid c_1 \rightarrow c'_1 \in \mathcal{M}\},$$

$$\mathcal{M}_{k+1} = \{c \rightarrow c' \in E_K \mid c, c' \in \text{Crit}(\bigcup_{i=1}^k \mathcal{M}_i), c_{k+1} \rightarrow c'_{k+1} \in \mathcal{M}\},$$

for $k = 1, \dots, m-1$. Define \mathcal{M}_K as the union $\mathcal{M}_K = \bigcup_{k=1}^m \mathcal{M}_k$. By construction, \mathcal{M}_K is a matching.

Proposition 3.3. *The matching \mathcal{M}_K is acyclic.*

Proof. For each cell $c = c_1 \times \cdots \times c_m$ we denote by $\ell(c)$ the sum of the dimension of the c_i plus the number of indices i for which c_i is a plus-cell.

Claim: The function $\ell(\cdot)$ is weakly decreasing on a directed path in $G^{\mathcal{M}}$.

Proof of Claim. Let $e = c \rightarrow c'$ be an edge in $G^{\mathcal{M}}$ for $c = c_1 \times \cdots \times c_m$ and $c' = c'_1 \times \cdots \times c'_m$. Then there is a unique index i for which $c_i \neq c'_i$.

Case 1: e is an edge from G . Then we have $\dim c_i = \dim c'_i + 1$. Hence $\ell(c) = \ell(c') + 1$ if the signs of c_i and c'_i coincide and $\ell(c) = \ell(c') + 2$ if the sign of c_i is positive and the sign of c'_i is negative.

Case 2: e is not an edge from G . Then we have $\dim c_i = \dim c'_i - 1$ and c_i is a plus-cell and c'_i is either a minus-cell or a e_\bullet^n . Thus $\ell(c) = \ell(c')$. \square

Thus on a directed cycle $p : c^1 \rightarrow \cdots \rightarrow c^r = c^1$ we must have $\ell(c^1) = \cdots = \ell(c^{r-1})$. From the proof of the claim it follows that on a directed cycle one can have only edges e which increase the dimension of the cells. But this cannot lead to a directed cycle and hence the matching is acyclic. \square

Since from now on we will only consider the matching \mathcal{M}_K defined above we write $\text{Crit}(K)$ for $\text{Crit}(\mathcal{M}_K)$. In addition, we write Crit_k for $\text{Crit}(K|_{[k]})$, $k = 1, 2, \dots, m$ and $\text{Crit}_{k\bullet}$ for $\text{Crit}(\text{link}_{K|_{[k+1]}}(k+1))$. The following proposition gives an inductive construction of the set of \mathcal{M}_K -critical cells. We use the following convention, if \mathcal{C} is a set of cells then we denote by \mathcal{C}^- the set of cells $c \times e_-^0$ for $c \in \mathcal{C}$, by \mathcal{C}^+ the set of cells $c \times e_+^{n-1}$ for $c \in \mathcal{C}$ and by \mathcal{C}^\bullet the set of cells $c \times e_\bullet^n$ for $c \in \mathcal{C}$.

Proposition 3.4. (i)

$$\text{Crit}_1 = \begin{cases} \{e_-^0\}, & \{1\} \in K \\ \{e_-^0, e_+^{n-1}\}, & \{1\} \notin K \end{cases}.$$

(ii) For $k \in [m-1]$

$$\text{Crit}_{k+1} = \text{Crit}_k^- \cup \text{Crit}_k^+ \cup \text{Crit}_{k\bullet}^\bullet \setminus \left(\text{Crit}_k^\bullet \cup \text{Crit}_{k\bullet}^+ \right).$$

In particular, any critical cell is a product of cells from the set $\{e_-^0, e_+^{n-1}, e_\bullet^n\}$.

Proof. The first claim follows from a simple inspection.

Let $k \in [m-1]$ and $c_1 \times \cdots \times c_{k+1} \in \text{Crit}_{k+1}$ be a critical cell. Set $c = c_1 \times \cdots \times c_k$.

- ◇ $c_{k+1} \neq e_\bullet^n$: Then by our inductive construction it follows that $c \in \text{Crit}_k$.
- ◇ $c_{k+1} = e_\bullet^n$. Then again the inductive construction implies that any matching of cells in $\mathcal{Z}_{K|_{[k]}}(D^n, S^{n-1})$ that is not induced by matching a cell $c_i = e_+^{n-1}$ for some $1 \leq i \leq k$ with e_\bullet^n can also be applied to $c \times c_{k+1}$.

It follows that $c_i \in \{e_-^0, e_+^{n-1}, e_\bullet^n\}$ for $1 \leq i \leq k$.

Assume that $c_{k+1} \notin \{e_-^0, e_+^{n-1}, e_\bullet^n\}$. Then there is a cell $c'_{k+1} \notin \{e_-^0, e_+^{n-1}, e_\bullet^n\}$ such that c_{k+1} and c'_{k+1} are matched in \mathcal{M} . Since $\text{supp}(c \times c_{k+1}) = \text{supp}(c \times c'_{k+1})$ it follows that $c' \times c'_{k+1}$ is a cell from $\mathcal{Z}_{K|_{[k+1]}}(D^n, S^{n-1})$. By the inductive nature of our matching the cells $c \times c_{k+1}$ and $c \times c'_{k+1}$ are matched in $\mathcal{M}_{K|_{[k+1]}}$. But this contradicts $c \times c_{k+1} \in \text{Crit}_{k+1}$. Hence we have $c_{k+1} \in \{e_-^0, e_+^{n-1}, e_\bullet^n\}$ and we distinguish the three cases.

- ◇ $c_{k+1} = e_-^0$.

From $c \in \text{Crit}_k$ and the nature of \mathcal{M} it follows that the cell $c \times c_{k+1}$ is not matched in \mathcal{M}_K . In particular, we have $c \times c_{k+1} \in \text{Crit}_k^- \subseteq \text{Crit}_{k+1}$.

- ◇ $c_{k+1} = e_+^{n-1}$.

Then from $c \in \text{Crit}_k$ it follows that $\text{supp}(c) \cup \{k+1\} \notin K|_{[k+1]}$. Hence $\text{supp}(c) \notin \text{link}_{K|_{[k+1]}}(k+1)$. This implies $c \times c_{k+1} \in \text{Crit}_k^+ \setminus \text{Crit}_{k\bullet}^+ \subseteq \text{Crit}_{k+1}$.

- ◇ $c_{k+1} = e_\bullet^n$.

Then for $c'_{k+1} = e_+^{n-1}$ we have $c \times c'_{k+1}$ is a cell in $\mathcal{Z}_{K|_{[k+1]}}(D^n, S^{n-1})$. Thus by definition of \mathcal{M} in order to have $c \times c_{k+1} \in \text{Crit}_{k+1}$ we need $c \times c'_{k+1} \notin \text{Crit}_{k+1}$. The latter implies $c \notin \text{Crit}_k$. Hence $c \times c_{k+1} \in \text{Crit}_{k\bullet}^\bullet \setminus \text{Crit}_k^\bullet \subseteq \text{Crit}_{k+1}$.

Summarizing the three cases and observing that the set involved in any two of the three cases are mutually disjoint implies the assertion. \square

By [Proposition 3.4](#) we know that if $c_1 \times \cdots \times c_m$ is a critical cell then $c_i \in \{e_-^0, e_+^{n-1}, e_\bullet^n\}$ for $1 \leq i \leq m$. Hence from now on we can identify a critical cell with an m -tuple in $\{+, -, \bullet\}^m$. For a cell $c = c_1 \times \cdots \times c_m \in V_m$ and $\text{sgn} \in \{+, -, \bullet\}$ we define $c(\text{sgn}) = -\infty$ if $\text{sgn} \neq c_i$ for $1 \leq i \leq m$ and $c(\text{sgn}) = \min\{i \mid c_i = \text{sgn}\}$ otherwise. The following is a simple corollary of [Proposition 3.4](#).

Corollary 3.5. *Let $c = c_1 \times \cdots \times c_m \in \text{Crit}(K)$ be a critical cell and for $1 \leq i \leq m$ set $A_i = \text{supp}(c) \cap \{i+1, \dots, m\}$. Then:*

- (i) *For $1 \leq i \leq m$ we have $c_1 \times \cdots \times c_i \in \text{Crit}(\text{link}_K(A_i)|_{[i]})$. If $i \in \text{supp}(c)$ then $c_1 \times \cdots \times c_{i-1} \notin \text{Crit}(\text{link}_K(A_i)|_{[i-1]})$.*
- (ii) *If $c(\bullet) \neq -\infty$ then*
 - (a) $-\infty < c(+) < c(\bullet)$ and

- (b) if $c_i = \bullet$ for some $1 \leq i \leq m$ then there exists an $1 \leq j < i$ such that $c_j = +$, $\text{supp}(c) \cup \{j\} \notin K$ and $\text{supp}(c) \setminus \{i\} \cup \{j\} \in K$.
- (iii) If $c(+) = -\infty$ then $c = (-, \dots, -)$.
- (iv) If $c(+) \neq -\infty$ and $c_i = +$ for some $1 \leq i \leq m$ then we have $\text{supp}(c) \cup \{i\} \notin K$.

Proof. (i) Suppose there is a cell $c'_1 \times \dots \times c'_i$ matched with $c_1 \times \dots \times c_i$ in $\mathcal{M}_{\text{link}_K(A_i)|_{[i]}}$ then by the inductive construction of the matching, the cells c and $c' = c'_1 \times \dots \times c'_i \times c_{i+1} \times \dots \times c_m$ are matched in \mathcal{M}_K . But this contradicts the assumption that $c \in \text{Crit}(K)$. If $c_i = \bullet$ then it is an immediate consequence of [Proposition 3.4](#) that $c_1 \times \dots \times c_{i-1} \notin \text{Crit}(\text{link}_K(A_i)|_{[i-1]})$.

(iv) Suppose there is an $1 \leq i \leq m$ such that $c_i = +$ and $\text{supp}(c) \cup \{i\} \in K$. Then the cells $c_1 \times \dots \times c_i$ and $c_1 \times \dots \times c_{i-1} \times \bullet$ are matched in $\mathcal{M}_{\text{link}_K(A_i)|_{[i]}}$ which contradicts (i).

(ii) (a) Let i be $c(\bullet)$. Then (i) implies that $c_1 \times \dots \times c_{i-1} \notin \text{Crit}(\text{link}_K(A_i)|_{[i-1]})$. Thus $c_1 \times \dots \times c_{i-1} \neq (-, \dots, -)$. Therefore by definition of $i = c(\bullet)$ it follows that there is an index $1 \leq j \leq i-1$ such that $c_j = +$. In particular, we have $-\infty < c(+) < i = c(\bullet)$.

(b) Let i be such that $c_i = \bullet$. Then by (i) $c' := c_1 \times \dots \times c_{i-1}$ is not a critical cell of $\mathcal{M}_{\text{link}_K(A_i)|_{[i-1]}}$. Thus for some $1 \leq j < i$ such that $c_j = +$ we have $\text{supp}(c) \setminus \{i\} \cup \{j\} \in K$. The second assertion follows from (iv).

(iii) If $c(+) = -\infty$ then it follows from (ii)(a) that $c(\bullet) = -\infty$ and hence $c = (-, \dots, -)$. \square

Example 3.6. Let K be the 4-gon with set of facets $\{13, 14, 23, 24\}$. Then the set of \mathcal{M}_K -critical cells is equal to $\text{Crit}(K) = \{(-, -, -, -), (-, -, +, \bullet), (+, \bullet, -, -), (+, \bullet, +, \bullet)\}$, which form $\mathcal{Z}_K(D^n, S^{n-1}) = S^{2n-1} \times S^{2n-1}$.

Lemma 3.7. *Let K be a simplicial complex on ground set $[m]$ and $c = c_1 \times \dots \times c_m \in \mathcal{M}(K)$ a cell such that $c(+) \neq -\infty$ and $c_i = +$ if and only if $i = c(+)$. Let $I = \{c(+)\} \cup \text{supp}(c)$. Then $c \in \text{Crit}(K)$ if and only if $c_I \in \text{Crit}(K_I)$ and $K_I = 2^I \setminus \{I\}$.*

Proof. By [Corollary 3.5](#) (ii)(a) we can assume that $I = \{c(+)<i_1<\dots<i_k\}$.

\Rightarrow If $c \in \text{Crit}(K)$ then by [Corollary 3.5](#) (ii)(b) $I \notin K$ and $I \setminus \{i_j\} \cup \{c(+)\} \in K$ for $1 \leq j \leq k$. Since $\text{supp}(c) \in K$ by definition we have that $K_I = 2^I \setminus \{I\}$. A simple check shows that c_I is critical for K_I .

\Leftarrow If $K_I = 2^I \setminus \{I\}$. Distinguish $\#I = 1$ and $\#I > 1$. $\#I = 1$ then $I = \{c(+)\}$ and by [Proposition 3.4](#) c is critical in K_I if and only if $I \notin K$. Thus $K_I = \{\emptyset\} = 2^I \setminus \{I\}$ and again by [Proposition 3.4](#) c is critical in K . Now let $\#I \geq 1$. Assume that $c \notin \text{Crit}(K)$. Then by [Proposition 3.4](#) (ii) the cell $c = c_1 \times \dots \times c_{i_{k-1}}$ either lies in $\text{Crit}_{i_{k-1}}$ or does not lie in $\text{Crit}(\text{link}_K(i_k))$. In the first case $I \setminus \{i_k\} \notin K$. Which contradicts $K_I = 2^I \setminus \{I\}$. In the second case we must have $I \setminus \{i_k\} \in \text{link}_K(i_k)$ and hence $I \in K$. Again this contradicts $K_I = 2^I \setminus \{I\}$. Thus $c \in \text{Crit}(K)$ and we are done. \square

4. PROOF OF **THEOREM 1.1** AND **COROLLARY 1.3**

Let K be a simplicial complex over the ground set $[m]$ with vertex decomposable Alexander dual K° , such that $\{i\} \in K$ for all $i \in [m]$ and m is a shedding vertex. By **Theorem 3.1**, the n -sphere moment-angle complex $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to the space $\mathcal{Z}_K(D^n, S^{n-1})^{\mathcal{M}_K}$. We need to see how \mathcal{M}_K -critical cells are glued to form the space $\mathcal{Z}_K(D^n, S^{n-1})^{\mathcal{M}_K}$. We need some preparatory lemmas.

For a cell $c = c_1 \times \cdots \times c_m$ define

$$J(c) := \begin{cases} -\infty & \text{if } c(+) = -\infty \\ \max\{i \mid c_i = +\} & \text{if } c(\bullet) = -\infty < c(+) \\ \max\{i < c(\bullet) \mid c_i = +\} & \text{if } -\infty \neq c(\bullet) \end{cases}$$

Proposition 4.1. *Let K be a simplicial complex on vertex set $[m]$ with vertex decomposable Alexander dual K° . If $c = c_1 \times \cdots \times c_m \in \text{Crit}(K)$ then $\tilde{c} = \tilde{c}_1 \times \cdots \times \tilde{c}_m \in \text{Crit}(K)$ for*

$$\tilde{c}_i = \begin{cases} c_i & \text{if } c_i \in \{-, \bullet\} \text{ or } i = J(c) \\ - & \text{if } c_i = + \text{ and } i \neq J(c) \end{cases}.$$

Proof. We proceed by induction on m . If $m = 1$ then either $K = \{\emptyset\}$ or $K = \{\emptyset, \{1\}\}$. In the first case $\text{Crit}(K) = \{-, +\}$ and for $c \in \text{Crit}(K)$ we have $c = \tilde{c}$. In the second case $\text{Crit}(K) = \{-\}$ and again for $c = \tilde{c}$ for $c \in \text{Crit}(K)$.

Assume $m \geq 2$. In the following for $c \in \text{Crit}(K)$ for which $c(\bullet) \neq -\infty$ we set $m(c) := \max\{i \mid c_i = \bullet\}$. We distinguish two cases:

$$\rightarrow c(\bullet) = -\infty$$

This condition is equivalent to $\text{supp}(c) = \emptyset$. Hence either $\tilde{c} = (-, \dots, -)$ or $\tilde{c} = (-, \dots, -, +, -, \dots, -)$ with $+$ in position $J(c)$. In the latter case we argue as follows. In the first case we clearly have $\tilde{c} \in \text{Crit}(K)$. Assume the second case. Since c is critical by **Proposition 3.4** we must have that $\{J(c)\} \notin K$. The latter implies that \tilde{c} is critical as well.

$$\rightarrow c(\bullet) \neq -\infty \text{ and } m(c) = m.$$

For this direction we make use of the notation used in **Proposition 3.4**

In this case **Proposition 3.4** implies that $(c_1, \dots, c_{m-1}) \in \text{Crit}(\text{link}_K(m))$. By induction it follows that $\tilde{c}_1 \times \cdots \times \tilde{c}_{m-1} \in \text{Crit}(\text{link}_K(m))$ and hence $\tilde{c} \in \text{Crit}_{m-1\bullet}^\bullet$. The construction of \tilde{c} allows to apply **Lemma 3.7** which shows that $N := \{J(c)\} \cup \text{supp}(c) \setminus \{m\} = \{J(\tilde{c})\} \cup \text{supp}(\tilde{c}) \setminus \{m\}$ is a minimal nonface of $\text{link}_K(m)$.

Assumption: $\tilde{c} \in \text{Crit}_{m-1}^\bullet$

\triangleleft We infer that $(\tilde{c}_1, \dots, \tilde{c}_{m-1}) \in \text{Crit}(K \setminus m)$. Again by **Lemma 3.7** we have that N is a minimal non-face of $K \setminus m$. Thus N is a minimal nonface of $\text{link}_K(m)$ that is also a minimal non-face of $K \setminus m$. This shows that $[m-1] \setminus N$ is a facet of $K^\circ \setminus m$ and $\text{link}_{K^\circ}(m)$. But this contradicts the fact that m is a shedding vertex of K° . Hence we have deduced a contradiction and the assumption is false. \triangleright

Now we know that $\tilde{c} \notin \text{Crit}_{m-1}^\bullet$ and $\tilde{c} \in \text{Crit}_{m-1\bullet}^\bullet$. Again **Proposition 3.4** shows that $\tilde{c} \in \text{Crit}(K)$.

$\rightarrow c(\bullet) \neq -\infty$ and $m(c) < m$. In this case by [Proposition 3.4](#) we have that $c_1 \times \cdots \times c_{m(c)} \in \text{Crit}(K \setminus m \setminus \cdots \setminus m(c) + 1)$. Since [Proposition 3.4](#) also shows that $\tilde{c} \in \text{Crit}(K)$ if and only if $\tilde{c}_1 \times \cdots \tilde{c}_{m(c)} \in \text{Crit}(K \setminus m \setminus \cdots \setminus m(c) + 1)$ the assertion follows by induction. \square

Corollary 4.2. *Let K be a simplicial complex with vertex decomposable Alexander dual K° . Then for $c \in \text{Crit}(K)$ with $c(\bullet) \neq -\infty$ the set $\{J(c)\} \cup \text{supp}(c)$ is a minimal non-face of K .*

Proof. Let \tilde{c} be as in [Proposition 4.1](#). Then [Proposition 4.1](#) implies that $\tilde{c} \in \text{Crit}(K)$. The construction of \tilde{c} assures that we can apply [Lemma 3.7](#). Hence $J(\tilde{c}) \cup \text{supp}(\tilde{c})$ is a minimal nonface of K . Since $J(c) = J(\tilde{c})$ and $\text{supp}(c) = \text{supp}(\tilde{c})$ by the construction of \tilde{c} the assertion follows. \square

Proposition 4.3. *Let K be a simplicial complex on ground set $[m]$ with vertex decomposable Alexander dual K° and let m be a shedding vertex for K° . Then for any $n \geq 1$ the inclusion $i : \mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \hookrightarrow \mathcal{Z}_{K \setminus m}(D^n, S^{n-1})$ is homotopically trivial.*

Proof. It is sufficient to find a contractible CW-subcomplex Y of the complex $\mathcal{Z}_{K \setminus m}$, which contains $\mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1})$. Let $c \in \text{Crit}(\text{link}_K(m))$. From [Corollary 4.2](#) we know that $N := \{J(c)\} \cup \text{supp}(c)$ is a minimal nonface of $\text{link}_K(m)$. Thus by [Remark 2.2](#) $[m-1] \setminus N$ is a facet of $(\text{link}_K(m))^\circ = K^\circ \setminus m$. Since m is a shedding vertex of K° it follows that $[m-1] \setminus N$ is not a face of $\text{link}_{K^\circ}(m) = (K \setminus m)^\circ$. Therefore $\{J(c)\} \cup \{\text{supp}(c)\}$ must be a face of $K \setminus m$. To $c = c_1 \times \cdots \times c_{m-1} \in \text{Crit}(\text{link}_K(m))$ we associate the cell $c^\bullet = c_1^\bullet \times \cdots \times c_{m-1}^\bullet$ defined by $c_i^\bullet = c_i$ if $i \neq J(c)$ and $c_i^\bullet = \bullet$ if $i = J(c)$. Consider

$$Y := \mathcal{Z}_{\text{link}_K(m)} \cup \left\{ c^\bullet \mid c \in \text{Crit}(\text{link}_K(m)) \setminus \{(-, \dots, -)\} \right\}.$$

Then it is easily checked that Y indeed is a subcomplex of $\mathcal{Z}_{K \setminus m}(D^n, S^{n-1})$. By construction the edge $c^\bullet \rightarrow c$ is in \mathcal{M}_Y for any critical cell $c \in \text{Crit}(\text{link}_K(m))$. Since $\mathcal{M}_{\text{link}_K(m)} \subseteq \mathcal{M}_Y$ it follows that $(-, \dots, -)$ is the only critical cell in Y and hence Y is contractible. \square

Before we can proceed to the proof of [Theorem 1.1](#) we have to recall some basic facts from homotopy theory. For the sake of completeness we provide proofs for facts we could not find an explicit reference for.

The first lemma is an elementary exercise in homotopy theory.

Lemma 4.4. (i) *If the pairs of CW-complexes (X, A) and (X', A') are relative homotopy equivalent, then X/A and X'/A' are homotopy equivalent.*
(ii) *The following homotopy equivalence holds:*

$$S^p \times S^q / \text{pt} \times S^q \simeq S^p \vee S^{p+q}.$$

(iii) *For any CW-complex X the quotient $X \times D^n / X \times \text{pt}$ is contractible.*

Proof. (i) Let $h : (X, A) \longrightarrow (X', A')$ be a relative homotopy equivalence. Then, the following diagram is commutative:

$$\begin{array}{ccccc} pt & \longleftarrow & A & \xrightarrow{i} & X \\ \downarrow & & \downarrow h|_A & & \downarrow h \\ pt & \longleftarrow & A' & \xrightarrow{i'} & X' \end{array}.$$

The statement follows from the Gluing Lemma (Lemma 2.4 in [22]).

- (ii) The quotient space $S^p \times S^q / pt \times S^q$ is a Thom space $\tau\epsilon_p$ of the trivial bundle ϵ_p over the sphere S^q , which is the one-point compactification $(S^q \times \mathbb{R}^p)^\wedge$. The lemma follows from the identity $(X \times Y)^\wedge = X^\wedge \wedge Y^\wedge$.
- (iii) The quotient space $A \times D^n / A \times pt$ is homotopy equivalent to the mapping cone of the inclusion $A \times pt \xrightarrow{i} A \times D^n$, which is homotopy trivial. Hence the quotient is contractible.

□

For the next lemma we need some basic facts from the theory of homotopy colimits. We refer the reader to [22] for results from that theory that are formulated in combinatorial language and for further references. The following lemma is a version of [14, Lem. 3.3].

Lemma 4.5. *Let $A \xrightarrow{i} X$ and $S \xrightarrow{j} D$ be homotopy trivial inclusions. Then the homotopy colimit space $Y = \operatorname{hocolim}\{X \times S \xleftarrow{i \times 1} A \times S \xrightarrow{1 \times j} A \times D\}$ is homotopy equivalent to the wedge of spaces*

$$Y \simeq X \times S / pt \times S \vee \Sigma(A \wedge S) \vee A \times D / A \times pt.$$

Proof. By the assumption for the maps i and j , we have

$$Y \simeq \operatorname{hocolim}\{X \times S \xleftarrow{pt \times 1} A \times S \xrightarrow{1 \times pt} A \times D\}.$$

Let $A * S = A \times S \times [0, 1] / \sim$ be the join of the spaces A and S , where $(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)$. The spaces A and S are embedded into the join by $i_2 : S \rightarrow A * S$ and $j_2 : A \rightarrow A * S$, where $i_2(y) = [(x, y, 1)], y \in S$ and $j_2(x) = [(x, y, 0)], x \in A$.

The space Y is homotopy equivalent to the colimit space

$$Y \simeq \operatorname{colim}\{X \times S \xleftarrow{i_1} S \xrightarrow{i_2} A * S \xleftarrow{j_2} A \xrightarrow{j_1} A \times D\},$$

where $i_1 : S \rightarrow X \times S$ and $j_1 : A \rightarrow A \times D$ are inclusions $i_1(y) = (pt, y), y \in S$ and $j_1(x) = (x, pt), x \in A$. Since there is a quotient map $p : A * S \rightarrow \Sigma(A \wedge S)$, which is a homotopy equivalence, we obtain

$$Y \simeq \operatorname{colim}\{X \times S \xleftarrow{j_2} S \xrightarrow{pt} \Sigma(A \wedge S) \xleftarrow{pt} A \xrightarrow{i_2} A \times D\}.$$

□

Now we are in position to provide the proof of **Theorem 1.1**.

Proof of Theorem 1.1. Wedge of Spheres: We prove by induction on the number m of vertices of the simplicial complex K that if K° is vertex decomposable then: $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres.

For $m = 1$ any simplicial complex has vertex decomposable Alexander dual and there is either no or exactly one critical cell different from $(-, \dots, -)$ in $\text{Crit}(K)$. Thus it follows immediately that $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to the wedge of spheres given by (1).

Assume $m \geq 2$. We first treat the case when K° is a simplex over some subset Ω of the ground set $[m]$. In this case $K = 2^\Omega * (2^{[m] \setminus \Omega} \setminus \{[m] \setminus \Omega\})$. Thus

$$\mathcal{Z}_K(D^n, S^{n-1}) = (D^n)^{m-r} \times ((D^n)^r \setminus \{(x_1, \dots, x_m) \in (D^n)^r \mid x_i \notin S^{n-1}\})$$

for $r = m - \#\Omega$. Thus $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a $(nr - 1)$ -sphere.

Next we consider the case when K° is vertex decomposable with shedding vertex m . The induction hypothesis asserts that for any simplicial complex L on $k < m$ vertices with vertex-decomposable Alexander dual simplicial complex the space $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres given by (1).

The space $\mathcal{Z}_K(D^n, S^{n-1})$ can be represented as a colimit of the following diagram

$$\begin{array}{ccc} & \mathcal{Z}_{\overline{\text{link}_K(m)}}(D^n, S^{n-1}) & \\ i \swarrow & & \searrow j \\ \mathcal{Z}_{\overline{K \setminus m}}(D^n, S^{n-1}) & & \mathcal{Z}_{\text{star}_K(m)}(D^n, S^{n-1}) \end{array}$$

where i and j are inclusions and for a simplicial complex L on $[m-1]$ we denote by \overline{L} the complex L considered as a simplicial complex over $[m]$. We have the following identities:

$$\begin{aligned} \mathcal{Z}_{\overline{K \setminus m}}(D^n, S^{n-1}) &= \mathcal{Z}_{K \setminus m}(D^n, S^{n-1}) \times S^{n-1}, \\ \mathcal{Z}_{\overline{\text{link}_K(m)}}(D^n, S^{n-1}) &= \mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \times S^{n-1}, \\ \mathcal{Z}_{\text{star}_K(m)}(D^n, S^{n-1}) &= \mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \times D^n \end{aligned}$$

Substituting in (4) and using the Projection Lemma it follows that $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to the homotopy colimit of the following diagram

$$\begin{array}{ccc} & \mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \times S^{n-1} & \\ i' \times \text{id} \swarrow & & \searrow \text{id} \times j' \\ \mathcal{Z}_{K \setminus m}(D^n, S^{n-1}) \times S^{n-1} & & \mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \times D^n \end{array}$$

where $i = i' \times \text{id}$, $j = \text{id} \times j'$ and i', j' are the inclusion maps. Clearly, the map j' is homotopy equivalent to the constant map. By Proposition 4.3 the map i' is also homotopy equivalent to the constant map. Hence Lemma 4.5 applies and shows that $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of the following three spaces:

$$(a) \left(\mathcal{Z}_{K \setminus m}(D^n, S^{n-1}) \times S^{n-1} \right) / \left(\text{pt} \times S^{n-1} \right)$$

By induction and [Lemma 2.3](#) we know that $\mathcal{Z}_{K \setminus m}(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres. Thus $\mathcal{Z}_{K \setminus m}(D^n, S^{n-1}) \times S^{n-1}$ is homotopy equivalent to a wedge of spheres times S^{n-1} by a homotopy that is the identity on $\text{pt} \times S^{n-1}$. But then there is a homotopy to a wedge of products of a sphere with S^{n-1} quotient by $\text{pt} \times S^{n-1}$. The latter is by [Lemma 4.4](#) (i) homotopy equivalent to a wedge of spheres,

$$(b) \Sigma \left(\mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \wedge S^{n-1} \right)$$

Assume there are ℓ elements from $[m]$ such that $\{\ell\}$ is not a face of $\text{link}_K(m)$. Then by induction and [Lemma 2.3](#) $\mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1})$ is homotopy equivalent to the product of $(S^{n-1})^\ell$ and a wedge of spheres given by (1). Since the smash product with S^{n-1} is the $(n-1)$ -fold suspension it follows that the space is homotopy equivalent to an n -fold suspension of a product of spheres and a wedge of spheres for some $n \geq 1$. Elementary homotopy arguments that show that it is homotopy equivalent to a wedge of spheres.

$$(c) \left(\mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \times D^n \right) / \left(\mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1}) \times \text{pt} \right).$$

This is contractible by [Lemma 4.4](#) (iii).

Hence $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres.

Counting the Spheres: In the second part of the proof we verify the exact formula for the number of spheres. Since we know that $\mathcal{Z}_K(D^n, S^{n-1})$ is homotopy equivalent to a wedge of spheres the rank of $\tilde{H}^i(\mathcal{Z}_K(D^n, S^{n-1}); \mathbb{K})$ counts the number of i -spheres in the wedge for any field \mathbb{K} .

In order to compute $\tilde{H}^i(\mathcal{Z}_K; \mathbb{K})$ we now invoke [Proposition 2.6](#). This immediately completes the proof for $n \geq 2$. For $n = 1$ the result follows from [12, Thm. 3.1] and the Hochster formula for the Betti numbers of the minimal free resolution of a monomial ideal (see [18] for the original proof or [9, p.226] for its statement). \square

Using the notation from [Proposition 3.4](#) we obtain the following corollary.

Corollary 4.6. *Let K be a simplicial complex on $[m]$ such that the Alexander dual complex K° is vertex-decomposable. Then*

(i) *If m is a shedding vertex of K° then*

$$\text{Crit}_m = \text{Crit}_{m-1}^- \cup \text{Crit}_{m-1}^+ \cup \text{Crit}_{m-1}^\bullet \setminus \{(-, \dots, -, +), (-, \dots, -, \bullet)\}.$$

(ii)

$$\begin{aligned} \tilde{H}^*(\mathcal{Z}_K(D^n, S^{n-1})) &\cong \tilde{H}^*(\mathcal{Z}_{K \setminus m}(D^n, S^{n-1})) \oplus \\ &\quad \tilde{H}^{*-n+1}(\mathcal{Z}_{K \setminus m}(D^n, S^{n-1})) \oplus \\ &\quad \tilde{H}^{*-n}(\mathcal{Z}_{\text{link}_K(m)}(D^n, S^{n-1})). \end{aligned}$$

(iii)

$$\mathcal{Z}_K(D^n, S^{n-1}) \simeq \bigvee_{c \in \text{Crit}(K)} S^{\dim(c)}.$$

Proof. (i) Suppose that $c \in \text{Crit}_{m-1} \cap \text{Crit}_{m-1\bullet}$. Then by [Corollary 4.2](#) we have that $N = \{J(c)\} \cup \text{supp}(c)$ is a minimal nonface of both $K \setminus m$ and $\text{link}_K(m)$. Thus $[m] \setminus N$ is a facet of both $K^\circ \setminus m$ and $\text{link}_{K^\circ}(m)$, which contradicts the condition that m is a shedding vertex of K° . We obtain that $\text{Crit}_{m-1} \cap \text{Crit}_{m-1\bullet} = \{(-, \dots, -)\}$ and the statement follows from [Proposition 3.4](#).

(ii) The assertion follows directly from homotopy decomposition given in the proof of [Theorem 1.1](#).

(iii) The assertion follows from (i) and (ii) by induction on the number of vertices using the fact that the Morse matching reduces accordingly. \square

Finally, we provide the proofs of [Corollary 1.2](#) and [Corollary 1.3](#).

Proof of Corollary 1.2. By [9, Prop. 8] we know that if $K^{(1)}$ is chordal then K° is vertex-decomposable. Since K is flag we have $\{i\} \in K$ for all $i \in [m]$. Hence the result follows from [Theorem 1.1](#). \square

Proof of Corollary 1.3. If we set $n = 2$ in [Proposition 2.6](#) then we obtain a homotopy equivalence

$$\Sigma \mathcal{Z}_K(D^n, S^{n-1}) \simeq \bigvee_{\substack{M \not\subseteq K \\ M \subseteq [m]}} \Sigma^{\#M+2} |K_M|.$$

Now the assertion follows from [Theorem 1.1](#). \square

Note that if [Proposition 2.6](#), respectively [1, Cor. 2.24], holds for $n \geq 1$ then one can strengthen [Corollary 1.3](#) to the assertion that the double suspension is homotopy equivalent to a wedge of spheres.

We finish with a few examples that illustrate our constructions and results.

Example 4.7. Let K be a simplicial complex with vertex decomposable dual on the vertex set $V = [6]$, such that the set of facets of $\text{link}_K(6)$ is $\{125, 134, 145, 234, 235\}$. The geometric realization $|\text{link}_K(6)|$ is the Möbius band. The Alexander dual complex $\text{link}_K(6)^\circ = K^\circ \setminus 6$ is the 5-gon $\{12, 13, 24, 35, 45\}$. Using [Proposition 3.4](#), we obtain $\text{Crit}(\text{link}_K(6)) = \{(-, -, -, -, -), (+, \bullet, \bullet, -, -), (+, \bullet, \bullet, +, -), (+, \bullet, -, \bullet, -), (+, \bullet, \bullet, -, +), (+, \bullet, \bullet, +, +), (+, \bullet, -, \bullet, +), (+, -, \bullet, -, \bullet), (+, -, \bullet, +, \bullet), (-, +, +, \bullet, \bullet), (-, -, +, \bullet, \bullet), (-, +, -, \bullet, \bullet)\}$. The necessary condition that 6 is the shedding vertex of K° is that there is no 1-simplex in $\text{link}_{K^\circ}(6) = (K \setminus 6)^\circ$. The minimal complex which satisfies this condition has to contain all 2-faces on the vertex set $[5]$ and hence $\binom{[5]}{3} \subset K \setminus 6$. This shows that $\{J(c)\} \cup \text{supp}(c) \in K \setminus 6$, for all $c \in \text{Crit}(\text{link}_K(6))$.

Example 4.8. Let K_1 and K_2 be simplicial complexes, which are proper subcomplexes of two simplices on disjoint ground sets. Since the n -sphere moment-angle complex corresponding

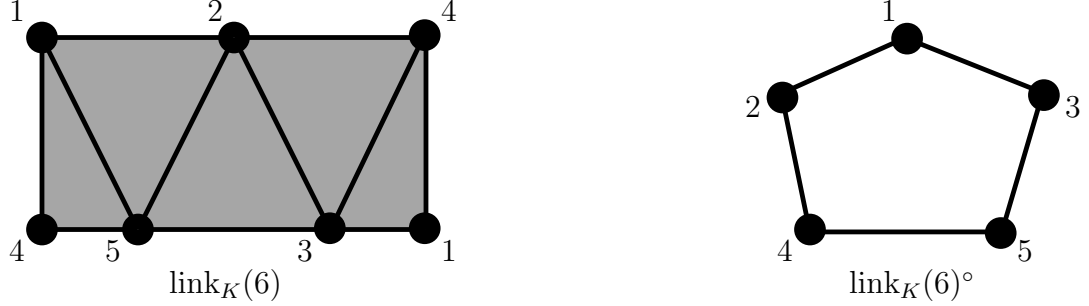


FIGURE 3. Complexes from Example 4.7

to the join $K_1 * K_2$ is the product $\mathcal{Z}_{K_1 * K_2}(X, A) = \mathcal{Z}_{K_1}(X, A) \times \mathcal{Z}_{K_2}(X, A)$, it follows from Theorem 1.1 that the dual complex of the join $(K_1 * K_2)^\circ$ is not vertex decomposable in general.

Example 4.9. Let K be the $(k-1)$ -skeleton of the simplex on m vertices. Then the dual complex K° is the $(m-k-1)$ -skeleton of the simplex on m vertices. It is easily seen that any skeleton of a simplex is vertex decomposable. Hence Theorem 1.1 applies. For any $M \subseteq [m]$ the complex K_M is the $(k-1)$ -skeleton of the full simplex with vertex set M . In particular, K_M is the full simplex if $\#M \leq k$. If $\#M > k$ then the homology of K_M is of rank $\binom{\#M-i}{k-1}$ in dimension $k-1$. If we apply Theorem 1.1 to our situation then we get the following homotopy equivalence which is also an immediate consequence of [14], [15]

$$\mathcal{Z}_K(D^n, S^{n-1}) \simeq \bigvee_{j=1}^{m-k} \binom{m}{k+j} \binom{k+j-j}{k} S^{nk+j(n-1)}.$$

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